Prime Merit
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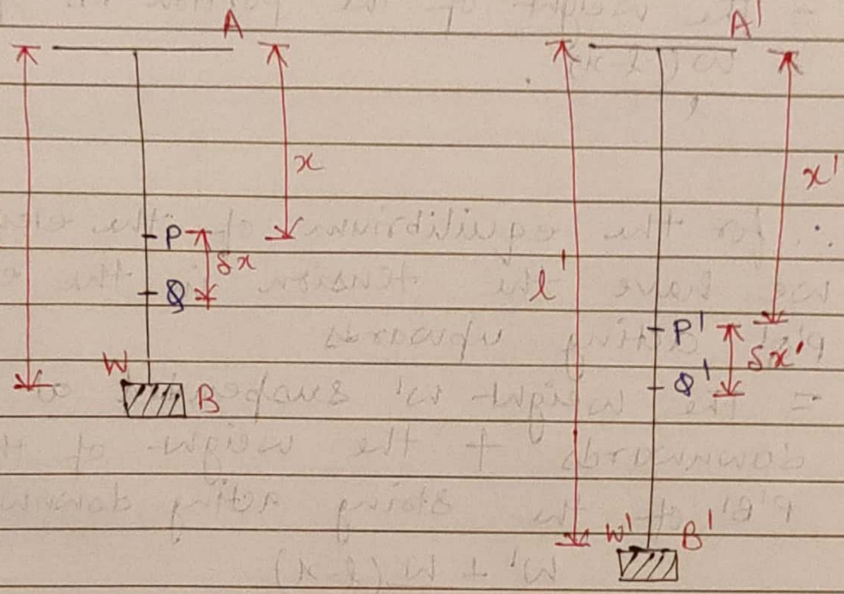
HEAVY ELASTIC STRING

Theorem:

To find the extension of a heavy elastic string of weight 'W' and natural length 'l' hanging from one end and supporting a weight W' at the other, where λ is the modulus of elasticity of the string.

Proof:



Let $AB = l$ be the natural length of heavy elastic string of weight W and $A'B'$ be its extended length when the weight W' is hanged on the other end B such that $A'B' = l'$.

Let us consider an small element $PQ = \delta x$ at a distance x from the point A whose position in the extended string is $P'Q' = \delta x'$ at a distance x' from A'.

Let T be the tension of the string about the depth x'.
Since, W be the weight of the string
Therfore, the weight per unit length of the string

$$= \frac{W}{l}$$

Now,
The weight of the portion $P'B'$ on the string
$=$ The weight of the portion PB of the string
$= \frac{W(l-x)}{l}$.

$\therefore$ for the equilibrium of the element $\delta x'$ we have the tension in the element $P'Q'$ acting upwards
$=$ The weight W' suspended at B' acting downwards $+$ the weight of the portion $P'B'$ of the string acting downwards

$$= W' + \frac{W}{l}(l-x)$$

By Hooke's law,

Tension T in $P'Q' = \lambda \frac{(\delta x' - \delta x)}{\delta x}$

Clearly,

$$W' + \frac{W}{l}(l-x) = \lambda \frac{(\delta x' - \delta x)}{\delta x}$$

$$= \lambda \left(\frac{\delta x'}{\delta x} - 1\right)$$

$$\Rightarrow \frac{w'l + w(l-x)}{\lambda l} = \frac{\delta x'}{\delta x} - 1$$

$$\Rightarrow \frac{w'l + w(l-x)}{\lambda l} + 1 = \frac{\delta x'}{\delta x}$$

$$\Rightarrow \frac{\delta x'}{\delta x} = \left\{1 + \frac{w'}{\lambda} + \frac{w}{\lambda l}(l-x)\right\}$$

$$\Rightarrow \delta x' = \left\{1 + \frac{w'}{\lambda} + \frac{w}{\lambda l}(l-x)\right\}\delta x$$

Integrating within the limits $x' = 0$ to l'
and $x = 0$ to l

$$\int_0^{l'} \delta x' = \int_0^{l} \left\{1 + \frac{w'}{\lambda} + \frac{w}{\lambda l}(l-x)\right\}\delta x$$

$$l' = l + \frac{w' l}{\lambda} + \frac{wl}{2\lambda}$$

$$l' - l = \frac{l}{\lambda}\left(w' + \frac{w}{2}\right).$$